Some implementation details

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October 3, 2003

1 The calculation of $\log 2$

1.1 Series

To calculate $\log 2$, we will use an expression of Sebah and Gourdon[1].

$$\log 2 = \frac{3}{4}\left(1 + \sum_{k=1}^{\infty}\prod_{\ell=1}^{k}\frac{-\ell}{8\ell+4}\right) \tag{1}$$

1.2 Absolute truncation error

Lemma 1. *Suppose we want to approximate* $\log 2$ *by*

$$l_2 = \frac{3}{4}\left(1 + \sum_{k=1}^{T}\prod_{\ell=1}^{k}\frac{-\ell}{8\ell+4}\right) \tag{2}$$

The absolute truncation error $E = |\log 2 - l_2|$ *is bounded by*

$$E \leq \frac{6}{7}\left(\frac{1}{8}\right)^{T+1} \tag{3}$$

Proof. From (1) and (2), we can bound the absolute truncation error E as follows:

$$
\begin{aligned}
|E| &= \left| \frac{3}{4} \sum_{k=T+1}^{\infty} \prod_{\ell=1}^{k} \frac{-\ell}{8\ell+4} \right| && (4) \\
&= \frac{3}{4} \left| \prod_{\ell=1}^{T+1} \frac{-\ell}{8\ell+4} \right| \left| \sum_{k=T+1}^{\infty} \prod_{\ell=T+2}^{k} \frac{-\ell}{8\ell+4} \right| && (5) \\
&< \frac{3}{4} \prod_{\ell=1}^{T+1} \frac{1}{8} \sum_{k=T+1}^{\infty} \prod_{\ell=T+2}^{k} \frac{1}{8} && (6) \\
&< \frac{3}{4} \left(\frac{1}{8}\right)^{T+1} \sum_{k=T+1}^{\infty} \left(\frac{1}{8}\right)^{k-T-1} && (7) \\
&= \frac{3}{4} \left(\frac{1}{8}\right)^{T+1} \frac{8}{7} = \frac{6}{7} \left(\frac{1}{8}\right)^{T+1} && (8)
\end{aligned}
$$

□

1.3 Reliable computation of $\log 2$

Suppose we want to calculate $\log 2$ such that the relative truncation error ε is bounded by $|\varepsilon| \leq |\overline{\varepsilon}|$. If we choose the approximant T such that

$$
\frac{6}{7} \left(\frac{1}{8}\right)^{T+1} \log 2 \leq |\overline{\varepsilon}| \tag{9}
$$

then we have

$$
|\varepsilon| = |E| \log 2 \leq \frac{6}{7} \left(\frac{1}{8}\right)^{T+1} \log 2 \leq |\overline{\varepsilon}| \tag{10}
$$

which is what we need. In order to satisfy (9), it is necessary that

$$
T \geq \frac{\log\left(\frac{6}{7}\log 2\right) - \log|\overline{\varepsilon}|}{\log 8} - 1 \tag{11}
$$

For the implementation, we will calculate an upperbound of T using interval arithmetic.

References

[1] Pascal Sebah and Xavier Gourdon. The logarithmic constant log(2). http://numbers.computation.free.fr/Constants/Log2/log2.html, september 2001.