# Error estimation for the implementation of arccot starting from arctan

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September 3, 2004

**Theorema 1.** *If the equation*

$$\operatorname{arccot} x = \arctan \frac{1}{x} \tag{1}$$

*is used to implement* $\operatorname{arccot} x$ *for* $x > 0$*, and the relative error of the the division and* arctan *implementation is bounded by* $|\epsilon| < \frac{4-\sqrt{2}}{4}$*, then the following holds for the calculated result* $\tilde{y}$*:*

$$\tilde{y} = (\operatorname{arccot} x)(1 + \delta) \tag{2}$$

*where* $|\delta| < 4|\epsilon|$*.*

*Proof.* Instead of calculating

$$\operatorname{arccot} x = \arctan \frac{1}{x} \tag{3}$$

an implementation will typically return

$$\tilde{f} = \arctan \left( \frac{1}{x}(1 + \epsilon_d) \right) (1 + \epsilon_a) \tag{4}$$

Where $\epsilon_d$ and $\epsilon_a$ are the relative errors introduced by the machine division and arctan. Both $|\epsilon_d|$ and $|\epsilon_a|$ are bounded by $|\epsilon|$.

We know that[1] $\arctan(z(1+\varepsilon))$ can be written as

$$\arctan(z(1+\varepsilon)) = \arctan(1+z)(1+\delta_0) \tag{5}$$

where

$$\delta_0 \leq \max_{|\eta|<|\varepsilon|} \left| \frac{1}{1+z^2(1+\eta)^2} \frac{z\varepsilon}{\arctan z} \right| \tag{6}$$

Because

$$\frac{z}{\arctan z} < z + 1 \tag{7}$$

[1]See the article on http://www.win.ua.ac.be/˜ jvvloet/onderzoek/properr.pdf

for $z > 0$, we have that

$$\delta_0 \leq \max_{|\eta|<|\varepsilon|} \left| \frac{z+1}{1+z^2(1+\eta)^2} \varepsilon \right| \tag{8}$$

We will now try to bound the factor $\varepsilon$ is multiplied by:

$$\frac{z+1}{1+z^2(1+\eta)^2} < 2 \tag{9}$$

To accomplish this, it is sufficient that for all real $z$

$$2z^2(1+\eta)^2 - z + 1 > 0 \tag{10}$$

Since expression (10) describes a concave[2] parabola, it suffices that it has no zeros, so its discriminant should be strictly negative:

$$1 - 8(1+\eta)^2 < 0 \tag{11}$$

The zeros of equation (11) are given by

$$\frac{\pm\sqrt{2} - 4}{4} \tag{12}$$

If $|\varepsilon| < \frac{4-\sqrt{2}}{4}$ and $|\eta| < |\varepsilon|$ , we have that (11) holds. And thus (10) is true as well. So from (9) and (8) we have that $|\delta_0| < |2\varepsilon|$. Applying (5) on (4) finally leads to

$$\tilde{f} = \left(\arctan \frac{1}{x}\right)(1+\delta_0)(1+\epsilon_a) \tag{13}$$

with $|\delta_0| < 2|\epsilon_d|$.

Because $|\epsilon_d| \leq |\epsilon|$ and $|\epsilon_a| \leq |\epsilon|$, it is not hard to see that equation (13) can be rewritten as

$$\tilde{f} = \left(\arctan \frac{1}{x}\right)(1+\delta) \tag{14}$$

with $|\delta| < 4|\epsilon|$, which completes the proof. □

[2]Is this the correct translation for 'hol'?